A geometric analysis of the Maxwell field in a vicinity of a multipole particle and new special functions

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Abstract

A method of solving Maxwell equations in a vicinity of a multipole particle (moving along an arbitrary trajectory) is proposed. The method is based on a geometric construction of a trajectory-adapted coordinate system, which simplifies considerably the equations. The solution is given in terms of a series, where a new family of special functions arises in a natural way. Singular behaviour of the field near to the particle may be analyzed this way up to an arbitrary order. Application to the self–interaction problems in classical electrodynamics is discussed.

1 Introduction

Classical approach to the problem of motion in classical electrodynamics is due to Dirac (cf. [3], [6]). In this approach, equations of motion of charged point particles, interacting *via* Maxwell field, are derived from field equations

by the following procedure: a generic state of the composed "particles + field" system is treated as a deformation of the "ground state" of the field, uniquely determined by the positions of the particles. The ground state is defined *via* a (non-local in time) decomposition of the actual field into the *retarded* (or *advanced*) field and the remaining "radiation field". Unfortunately, to decide what is *the retarded* or *the advanced* field, the entire trajectory of the particle must be known in advance and, whence, the causality of the theory is violated.

Recently, it was shown that such a non-causal procedure may be avoided. In this approach, equations of motion are a simple consequence of the conservation laws imposed on an (appropriately defined, "already renormalized") total four-momentum of the "particles + field" system (see e.g. [7], [5], [11]). Here, the "ground state" of the system may be defined *via* a conditional minimization of the energy, with the positions and the charges of the particles being fixed.

Mathematically, this leads to a simple (elliptic) variational problem for the behaviour of the field in a topologically non-trivial region of $\mathbb{R}^3$ (exterior of the particles), where the charges of the particles provide the necessary boundary conditions. Such a reduction of *electrodynamics* to *electrostatics* provides basic information about the behaviour of the field in the vicinity of the particle (e. g. the leading r^{-2} term for the monopole particle or the r^{-3} behaviour for the dipole) but fails to capture its more subtle features, which are necessary to describe such phenomena like polarization, which is necessary for the analysis of the dynamical stability of the classical electrodynamics (see e. g. discussion of these issues in [10] and the references therein).

A substantial improvement of description of the field singularity near the trajectory is obtained if we reduce the dynamics with respect to a one-parameter group of boosts (see [11]), instead of the one-parameter group of time translations, corresponding to the standard, flat "(3+1)-decomposition" of the Minkowski spacetime. As a result, we obtain the "electrostatics in a uniformly accelerated reference frame" as a proper tool to construct the ground state of the field. Here, field equations are again elliptic, but the flat Eucledean geometry is replaced by the Lobaczewski geometry. Within this framework, the Born solution describing Maxwell field attached to a uniformly accelerated particle (see e. g. [13]) arises as a fundamental solution of the Laplace-Beltrami operator Δ_L in the Lobaczewski space: $\Delta_L \psi = \delta_0$, where the right hand side is the Dirac delta distribution (a nice analysis of

this structure was proposed by Turakulov in [14]).

In the present paper we show how to use this (naturally arising) Lobaczewski three-geometry to the analysis of the full dynamical problem: analysis of the Maxwell field generated by a (moving along an arbitrary trajectory) multipole particle. Our method consists in splitting the portion of spacetime in the vicinity of a generic trajectory of a particle into a collection of 3-spaces orthogonal to the trajectory. Each of these spaces carries a natural Lobaczewski geometry, uniquely implied by the instantaneous acceleration of the particle. As a result of this construction, the flat spacetime geometry is replaced by a curved (general relativistic) geometry, whose parameters are uniquely defined by the parameters of the trajectory in question. We show that the Maxwell equations may be rewritten in terms of this geometry. This method provides a new, powerful tool, which enables us to solve Maxwell equations by a successive approximation method.

The results presented here give us an important improvement and generalization of earlier methods (see [8] or [9]), based on the use of the Fermi coordinates, defined by the Fermi tetrad attached to the trajectory. The geometric construction proposed in the present paper will be referred to as a *modified Fermi system*.

Naively speaking, the standard (3+1)-approach consists in approximating the generic trajectory by a straight line, matching only the instantaneous velocity of the particle. In the approach based on Fermi coordinates, we use much better approximation given by hyperbolic (uniformly accelerated) trajectories, which match not only the velocity but also the acceleration of the particle. The value of acceleration is encoded in the parameter of the Lobaczewski 3-space arising in this construction. In our new approach (which we refer to as the "modified Fermi"), proposed in the present paper, we use still better approximation: instantaneous acceleration $a(t)$ of the trajectory implies the parameter of each of the (Lobaczewski) 3-space folia $\{t = \text{const.}\}$ of the constructed geometry; there is, however, a non vanishing *shift vector* whose value encodes the derivatives $\dot{a}^j(t)$.

The basic technical tool of our method consists in deriving an explicit formula for the operator inverse to the Lobaczewski curl operator (which we denote by "curl_φ") on a certain space of divergence-free vector fields. The inversion requires a successive solution of Legendre differential equation with non trivial right hand side. This, in principle, could lead very quickly to non-elementary functions (see formula (33) in Section 4). To our great surprise, we were able to express the results (up to the order "+1" in the

radius r) in terms of rational functions of two universal combinations of the radial coordinate, namely $u = ar/2$ and $R = \log(\frac{1-u}{1+u})$, where a is the (scalar) acceleration of the particle (see Section 6). This result (together with the extremely simple and natural physical context) leads us to a conjecture that the functions obtained this way form a new family of special functions, which we discuss at the end of the paper.

Physically, our results form a basis for the theory of motion of a polarizable particle: a particle which carries a dynamical electric or magnetic moment, i. e. a moment which is not frozen within the particle but depends dynamically upon the surrounding electromagnetic field. There is a serious hope that such a theory would be free of the standard non-stability problems accompanying the Abraham-Dirac theory (see [10] and the discussion therein).

The paper is organized as follows. In Section 2 we define the modified Fermi system. In Section 3 we derive Maxwell equations in the modified Fermi system and interpret them in terms of the Lobaczewski geometry. In Section 4 we define basic fields $\mathbf{X}$ and $\mathbf{Y}$ and show that the operations div, curl_φ and $\#$ act on $\mathbf{X}$, $\mathbf{Y}$ in a simple way. These results give a generalization of the construction proposed in [9]. In Section 5 we solve Maxwell equations in a vicinity of the multipole particle, in terms of a series consisting of the $\mathbf{X}$ and $\mathbf{Y}$ fields. The procedure is analogous to that used in [9]. In case of the monopole and the dipole particle, the resulting series up to the order r^1 is explicitely given in Section 6, where r is the radial coordinate in the modified Fermi system. In Section 7 we propose a conjecture concerning the special functions arising naturally in our construction. Several specific computations and proofs are shifted to the Appendices A-D. The results of Sections 6 and 7 were obtained partially with help of the symbolic computations program MAPLE 8.

Thorough the paper we use greek indices (running from 0 to 3) to label space-time coordinates, and latin indices (runing from 1 to 3) to label space coordinates. We always use Einstein convention: summation over repeated indices (in both space-time and space cases). The components of an n-tensor $\mathbf{T}$ are denoted by $T_{i_1 i_2 \ldots i_n}$.

2 The Fermi–propagated and the modified Fermi–propagated systems

In this section we recall properties of the Fermi–propagated system of coordinates, in which the particle "remains at rest" at each instant of time. Next we modify that system in a way which is similar to transition from cartesian to bispherical (bipolar) coordinates in $\mathbb{R}^3$, with the particle's position being one of its centers (poles).

Let y^λ, $\lambda = 0, 1, 2, 3$, denote the (Minkowski) spacetime coordinates in a fixed inertial ("laboratory") system, corresponding to the metric tensor $\eta = \mathrm{diag}(-, +, +, +)$. Consider an arbitrary particle's trajectory $q^\lambda(t) = (t, q^k(t))$. Let $\tau = \tau(t)$ denote a particle's proper time along the trajectory. The normalized four-velocity is given by $\mathbf{u} = dq/d\tau$ and the particle's acceleration by $\mathbf{a} = d\mathbf{u}/d\tau = d^2q/d\tau^2$. We define (see e.g. [11]) the rest–frame space $\Sigma_{\tau(t)}$ as the hyperplane orthogonal to the trajectory (i.e. to the vector $\mathbf{e}_{(0)} := \mathbf{u}$) at the point $q(t)$. We choose an orthonormal basis $\mathbf{e}_{(l)}$, $l = 1, 2, 3$, in Σ_τ, such that $\mathbf{e}_{(\mu)}$ are positively oriented. Thus $(\mathbf{e}_{(\alpha)}|\mathbf{e}_{(\beta)}) = \eta_{\alpha\beta}$. Since $(\mathbf{u}|\mathbf{a}) = 0$, one gets $\mathbf{a} = a^l\mathbf{e}_{(l)}$ for some a^l.

At each instant of time τ, the above system may be subject to the $SO(3)$ group of rotations of the *Dreibein* $(\mathbf{e}_{(l)}(\tau))$, playing role of the group of gauge transformations. The Fermi-propagated system (see e.g. [11]) is defined by the following condition:

$$d\mathbf{e}_{(l)}/d\tau = a^l\mathbf{u} \ .$$

Once the above condition is satisfied, there remains a single, global (time-independent) $SO(3)$ gauge transformation, corresponding to an arbitrary choice of initial conditions: $\mathbf{e}_{(l)}(\tau_0)$. Moreover, we obtain in this case:

$$d\mathbf{u}/d\tau = a^l\mathbf{e}_{(l)} \ ,$$

which enables us to interpret a^l, $l = 1, 2, 3$; (with $a^0 = 0$) as components of the acceleration vector. Next, we define in a neighbourhood of the trajectory the Fermi-propagated (local) coordinates $\xi^\mu := (\xi^0, \xi^l)$ putting $\xi^0 := \tau$ and

$$y^\lambda = q^\lambda(\tau) + \xi^l e^\lambda_{(l)}(\tau). \tag{1}$$

Hence, ξ^l are cartesian coordinates on Σ_τ, related to $\mathbf{e}_{(l)}$ and centered at the particle's position ($\xi^l = 0$ for $y = q(t)$). The Minkowski metric tensor γ in that system is given by

$$\gamma_{kl} = \delta_{kl}, \quad \gamma_{0l} = 0, \quad \gamma_{00} = -N^2, \tag{2}$$

where $k, l = 1, 2, 3$, $N = 1 + a_l \xi^l$ (see e.g. [11], p. 373, but with ξ denoted there by x).

Now, we change variables on each Σ_τ and define the "modified Fermi-propagated" system $x^\mu = (x^0, x^l)$ by putting $x^0 := \tau = \xi^0$ and:

$$x^l = \frac{\xi^l + \frac{1}{2} a^l \rho^2}{1 + a_i \xi^i + \frac{1}{4} a^2 \rho^2}, \tag{3}$$

where $a = (\gamma_{kl} a^k a^l)^{1/2}$, $\rho = (\gamma_{kl} \xi^k \xi^l)^{1/2}$. It is easy to check that the inverse transformation is given by the following formula:

$$\xi^l = \frac{x^l - \frac{1}{2} a^l r^2}{1 - a_i x^i + \frac{1}{4} a^2 r^2}, \tag{4}$$

where $r = (\gamma_{kl} x^k x^l)^{1/2}$ ($a_i = a^i$ is the same as before because the particle is situated at $x = \xi = 0$, where $\partial x^k / \partial \xi^l = \delta^k_l$ and (2) holds).

A simple computation shows that the flat (Minkowski) metric tensor has now the following components:

$$g_{kl} = \frac{N^2}{\varphi^2} \delta_{kl}, \quad g_{0l} = \frac{\partial \xi^k}{\partial \tau} \frac{\partial \xi^k}{\partial x^l}, \quad g_{00} = -N^2 + \frac{\partial \xi^k}{\partial \tau} \frac{\partial \xi^k}{\partial \tau}, \tag{5}$$

with

$$\varphi = 1 - (ar/2)^2 \tag{6}$$

Of course, we have: $g^{00} = g(d\tau\, ; d\tau) = -N^{-2}$, and thus the lapse function N is the same as for the previous Fermi system, because the time variable has not been changed (cf. [9]). In terms of the new coordinates, its value may be written as follows: $N = \varphi/M$, where we denote $M = 1 - a_i x^i + \frac{1}{4} a^2 r^2 = (1 + a_k \xi^k + \frac{1}{4} a^2 \rho^2)^{-1}$. Moreover, $\sqrt{\det g_{kl}} = (N/\varphi)^3 = M^{-3}$. Denoting by $\tilde{g}$ the 3-dimensional inverse of (g_{kl}), $k, l = 1, 2, 3$, the shift vector may be calculated as follows:(cf. [9])

$$N^k = \tilde{g}^{kl} g_{0l} = \frac{\partial x^k}{\partial \xi^s} \frac{\partial \xi^s}{\partial \tau} = x^k \dot{a}_j x^j - \frac{1}{2} r^2 \dot{a}^k, \tag{7}$$

where "dot" denotes the derivative w.r.t. τ, e.g. $\dot{a}_j = da_j/d\tau$.

To illustrate the geometric structure of the above coordinate system, let us apply to the above picture such a rotation $\mathcal{O}$, which positions the third axis $\mathbf{e}_{(3)}$ in the direction of the acceleration $\mathbf{a}$, i. e. such that we have: $\mathbf{a} = a\, \mathbf{e}_{(3)}(\tau)$.

Apply now rotation $\mathcal{O}$ to coordinates (x^1, x^2, x^3) and denote by (z^1, z^2, z^3) new coordinates on Σ_τ, obtained *via* such a rotation. Next, using z^k as Cartesian coordinates, construct the corresponding spherical coordinates (r, η, ϕ). Finally, define the variable μ by

$$r = \frac{2}{a}\exp(-\mu).$$

Then, it is easy to show that (μ, η, ϕ) form the bispherical system of coordinates on the Euclidean space Σ_τ, with the particle's position being one of its centers: $r = 0$ (i.e. $\mu \to +\infty$). The vector connecting this center with the other center of the bispherical system is parallel but opposite to the acceleration and its length equals $\frac{2}{a}$.

3 Maxwell equations

Maxwell equations: $\mathrm{d}f = 0$ and $\mathrm{d}{*f} = \mathcal{J}$, with $*$ denoting the Hodge "star" operator, can be written in an arbitrary system of coordinates, in an arbitrarily curved spacetime, as follows (cf. e.g. [9]):

$$\partial_{[\gamma} f_{\mu\nu]} = 0, \tag{8}$$

$$\partial_\nu \mathcal{F}^{\mu\nu} = \mathcal{J}^\mu, \tag{9}$$

$$\mathcal{F}^{\mu\nu} = \sqrt{-\det g}\; g^{\mu\alpha} g^{\nu\beta} f_{\alpha\beta}\ , \tag{10}$$

where the brackets "[ ]" denote the complete anti-symmetrization. Moreover, we denote: $\mathcal{J}^\mu = \sqrt{-\det g} J^\mu$, where J^μ is the four–current vector and $\mathcal{J}^\mu$ is the current density (an "odd three-form", see e. g. [2]), satisfying the continuity equation $\partial_\mu \mathcal{J}^\mu = 0$. The two vector–densities: $\mathcal{D}$ (electric induction) and $\mathcal{B}$ (magnetic induction), are defined (cf. [9]) as the following components of these tensors: $\frac{1}{2} f_{kl}\epsilon^{klm} = \mathcal{B}^m$, $\mathcal{F}^{0k} = \mathcal{D}^k$, where ϵ^{klm} is the standard Levi–Civitta symbol (totally antisymmetric and normalized: $\epsilon^{123} = 1$). Due to (5), the covariant components of the magnetic field, calculated in our modified-Fermi coordinates (3), are equal to (cf. [9]): $B_m = g_{mk} B^k = g_{mk} (\det g_{kl})^{-1/2} \mathcal{B}^k = (\varphi/N)\mathcal{B}^m$. Hence, we have: $N B_m = \varphi \mathcal{B}^m$ and, similarly, $N D_m = \varphi \mathcal{D}^m$. These formulae show that a substantial simplification of the field equations is obtained, if we use a fictitious, flat metric δ_{kl} to identify "upper and lower indices", i. e. to fix an isomorphism between vectors and covectors. From now on, we strictly observe this

convention. Consequently, we can rewrite Maxwell equations (8)–(10) (cf. (9)-(12) of [9]) as equations for the two quantities $(\mathcal{D}, \mathcal{B})$, treated as two vectors in this fictitious flat geometry:

$$\partial_k \mathcal{D}^k = \mathcal{J}^0, \tag{11}$$

$$\partial_k \mathcal{B}^k = 0, \tag{12}$$

$$\dot{\mathcal{D}}^k - \partial_l (N^l \mathcal{D}^k - N^k \mathcal{D}^l) = \epsilon^{kil} \partial_i (\varphi\, \delta_{lj} \mathcal{B}^j) - \mathcal{J}^k, \tag{13}$$

$$\dot{\mathcal{B}}^k - \partial_l (N^l \mathcal{B}^k - N^k \mathcal{B}^l) = -\epsilon^{kil} \partial_i (\varphi\, \delta_{lj} \mathcal{D}^j), \tag{14}$$

where the vector N^k is uniquely implied by the derivative of the trajectory up to the third order, according to formula (7), whereas the "conformal factor" φ is implied by the second derivative, according to formula (6). We introduce the following short-hand notation for the differential operators appearing here:

$$\mathbf{W}^{\#} = \dot{\mathbf{W}} + \tilde{\mathbf{W}}, \tag{15}$$

where

$$\tilde{W}^k = \partial_l (N^k W^l - N^l W^k) \ , \tag{16}$$

and

$$\mathrm{curl}_\varphi \mathbf{W} := \mathrm{curl}(\varphi \mathbf{W}) \ .$$

This enables us to rewrite Maxwell equations as follows:

$$\mathrm{div} \mathcal{D} = \mathcal{J}^0, \tag{17}$$

$$\mathrm{div} \mathcal{B} = 0, \tag{18}$$

$$\mathcal{D}^{\#} = \mathrm{curl}_\varphi \mathcal{B} - \mathcal{J}, \tag{19}$$

$$\mathcal{B}^{\#} = -\mathrm{curl}_\varphi \mathcal{D} \ . \tag{20}$$

Now, we are going to show that the differential operator "curl_φ" can be nicely interpreted as a genuine curl operator in the Lobaczewski geometry. For this purpose consider the covectors which are subject to differentiation in the definition of curl_φ, see formulae (13), (14):

$$d_l := \varphi\, \delta_{lj} \mathcal{D}^j \ , \quad b_l := \varphi\, \delta_{lj} \mathcal{B}^j \ .$$

We want to find an appropriate (yet another) 3-metric μ on each leaf Σ_τ, such that the above covectors (differential one-forms) are related with the

vector densities $\mathcal{D}$ and $\mathcal{B}$ (differential three-forms) *via* the standard Hodge formula:

$$\mathcal{D}^k = \sqrt{\mu}\,\mu^{kl} d_l \ , \quad \mathcal{B}^k = \sqrt{\mu}\,\mu^{kl} b_l \ . \tag{21}$$

It is easily seen that μ must be conformally similar to the flat metric: $\mu_{kl} = \lambda\delta_{kl}$. Taken into account that $\sqrt{\mu}\,\mu^{kl} = \lambda^{\frac{1}{2}}\delta^{kl}$, we conclude that to fulfil (21) we must take $\varphi\lambda^{\frac{1}{2}} = 1$, i. e. $\lambda = \varphi^{-2}$. It is a matter of easy calculations to check that, indeed, the resulting metric tensor:

$$\mu_{kl} := \varphi^{-2}\delta_{kl} = \frac{1}{1-(ar/2)^2}\ \delta_{kl}\ , \tag{22}$$

defined on the ball $K := \{\vec{x} \in \mathbb{R}^3 : r = \|\vec{x}\| < \frac{2}{a}\}$ is equal to the Lobaczewski metric with constant negative curvature $R = -6a^2$.

In the particular case of a uniformly accelerated trajectory, i. e. when $\dot{a} = 0$, the shift vector vanishes: $N = 0$. As noticed by several authors, the "electrostatics" (with respect to such a uniformly accelerated reference system!) implies: $\mathrm{curl}_\varphi \mathcal{D} = 0$, equivalent to $d = \mathrm{grad}\ \psi$. Then, Gauss equation (17) reduces such a theory to the standard "potential theory" in the Lobaczewski space: $\mathrm{div\ grad}\ \psi = \Delta_L \psi = \mathcal{J}^0$. The fundamental solution of this equation, corresponding to the point particle: $\mathcal{J}^0 = \delta_0$ (Dirac-delta-like charge density – cf. [14]), describes the electromagnetic field accompanying the uniformly accelerated particle and is equal to the Born solution (cf. [1] or [13])

4 The calculus of X and Y fields

In this section we define elementary fields $\mathbf{X}$ and $\mathbf{Y}$, which enable us to reduce the asymptotic analysis of Maxwell equations in a neighbourhood of a moving, polarized particle, to a relatively simple algebra of these fields.

Given a pair (n, m) of natural numbers such that $n = 1, 2, \ldots$, $m = 0, 1, 2, \ldots$, consider the vector space $L_{n,m}$ of Laurent series of a single variable r, of the form

$$\sum_{k=0}^{\infty} c_k r^{-n+2m+2k},$$

convergent in an annular neighbourhood $0 < r < \epsilon$, where the coefficients c_k are real. Observe that we have $L_{n,m+1} \subset L_{n,m}$. By the order of the series we

understand the order of its first non-vanishing term (i. e. f is of order l if it behaves at $r=0$ like r^l).

For any pair $(f,\mathbf{Q})$, where $f\in L_{n0}$, and $\mathbf{Q}=(Q_{i_1\cdots i_n})$ is a completely symmetric, traceless tensor of rank n (i. e. satisfying: $\delta^{i_1i_2}Q_{i_1i_2\cdots i_n}=0$), consider the following vector fields:

$$X^k_{(n)}(f,\mathbf{Q})=[(n+1)f-rf_{,r}]\frac{1}{r^{n+3}}x^kQ_{i_1\cdots i_n}x^{i_1}\cdots x^{i_n}+f_{,r}\frac{1}{r^n}Q^k_{i_2\cdots i_n}x^{i_2}\cdots x^{i_n}, \tag{23}$$

$$Y^k_{(n)}(f,\mathbf{Q})=\frac{f}{\varphi}\frac{1}{r^{n+1}}\epsilon^{kjl}x_jQ_{li_2\cdots i_n}x^{i_2}\cdots x^{i_n}\ . \tag{24}$$

For an exceptional case $n=0$, every $\mathbf{Q}$ is a scalar and, therefore, the field $Y_{(0)}$ is not defined. Moreover, we assume in this case that f is constant.

This is an appropriate generalization of the fields X and Y, which were defined in [9] for purposes of the analysis of the Maxwell equations in terms of the standard Fermi system. As will be seen in the sequel, this generalization enables us to describe the asymptotic behaviour of the Maxwell field in a neighbourhood of a polarized, point particle in a much efficient way.

It can be easily shown (cf. Appendix A) that, outside of the center $r=0$, the fields X and Y are divergence-free:

$$\mathrm{div}\mathbf{X}_{(n)}(f,\mathbf{Q})=\mathrm{div}\mathbf{Y}_{(n)}(f,\mathbf{Q})=0, \tag{25}$$

and the operator curl_φ acts in a simple way on these fields:

$$\mathrm{curl}_\varphi\mathbf{Y}_{(n)}(f,\mathbf{Q})=-\mathbf{X}_{(n)}(f,\mathbf{Q})\ , \tag{26}$$

$$\mathrm{curl}_\varphi\mathbf{X}_{(n)}(f,\mathbf{Q})=-a^2\mathbf{Y}_{(n)}(h_{(n)}(f),\mathbf{Q}), \tag{27}$$

where

$$h_{(n)}(f)=-\frac{\varphi^2}{a^2}f_{,rr}+\frac{\varphi r}{2}f_{,r}+\frac{n(n+1)\varphi^2}{a^2r^2}f\ . \tag{28}$$

(The full action of these operators understood in the sense of distributions contains also Dirac-δ-like terms at $r=0$, cf. Appendix D). A much simpler form of the operator $h_{(n)}$ is obtained if we introduce the following new variables: $u=ar/2$, $z=\frac{1}{2}(u^{-1}+u)=\frac{1}{2}(\frac{2}{ar}+\frac{ar}{2})=\cosh\mu$. Then $z\geq 1$ and assuming $u<1$ (i. e. near to the particle) we have: $r=2u/a=\frac{2}{a}(z-\sqrt{z^2-1})$. Now, $h_{(n)}$ takes the form:

$$h_{(n)}[g(z)]=(z^2-1)\{(1-z^2)\frac{d^2g}{dz^2}-2z\frac{dg}{dz}+n(n+1)g\}. \tag{29}$$

Equation $h_{(n)}[g(z)] = 0$ is equivalent to the Legendre equation and, therefore, has two independent solutions: the Legendre polynomial of order n: $P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n = A_n(r)$ and the Legendre function $\frac{1}{2}P_n(z)\log(\frac{z+1}{z-1})+v_{n-1}(z) = B_n(r)$, where v_{n-1} is a polynomial of order $n-1$, $v_{-1} = 0$, P_n and v_n are even (odd) for n even (odd) (cf. [15], pp. 302 and 319). Thus $A_n(r)$ and $B_n(r)$ are solutions of equation $h_{(n)}(f) = 0$. A_n has the same parity as n, B_n – the opposite parity (one can formally define them for $|u| = \frac{a|r|}{2} < 1$).

It can be proved (see Appendix B) that A_n is a Laurent polynomial of r with the lowest power r^{-n}, whereas B_n is regular at $r = 0$ and its Taylor series starts from r^{n+1}.

Proposition. There exists a unique operator (the index m is suppressed for simplicity) $s_{(n)} : L_{nm} \to L_{n,m+1}$, which is inverse to $h_{(n)} : L_{n,m+1} \to L_{nm}$.

Proof. Because solutions of the homogeneous problem are known, equation

$$h_{(n)}(f) = l \tag{30}$$

can be solved by the "variation of constants" method:

$$f(r) = A_n(r)\int \frac{a^2 l(r) B_n(r)\mathrm{d}r}{\varphi^2(A_n {B_n}' - B_n {A_n}')} - B_n(r)\int \frac{a^2 l(r) A_n(r)\mathrm{d}r}{\varphi^2(A_n {B_n}' - B_n {A_n}')} \tag{31}$$

where the Wronskian $A_n {B_n}' - B_n {A_n}'$ is proportional to φ^{-1}, so the denominators under integrals are proportional to φ. Assume now that $l \in L_{nm}$. We define $s_{(n)}(l)$ as the right hand side of (31) with the integrals performed term by term, using the formula $\int r^\alpha \mathrm{d}r = \frac{r^{\alpha+1}}{\alpha+1}$ $(\alpha \neq -1)$. Then $s_{(n)}$ is linear. Moreover, the Laurent series of the function under the first integral is odd and starts at least from r^{1+2m}, so the series for the integral is even and starts at least from r^{2+2m}. The Laurent series under the second integral is even and starts at least from r^{-2n+2m}. Its integral is odd and starts from $r^{-2n+2m+1}$. (This proves that, indeed, the case $\alpha = -1$ never occurs.) Thus, $s_{(n)}(l)$ is even or odd (depending upon n) and starts at least from $r^{-n+2(m+1)}$, i. e. we have $s_{(n)}(l) \in L_{n,m+1}$. Any other solution f of (30) differs from $s_{(n)}(l)$ by a combination of A_n and B_n. But the condition $f \in L_{n,m+1}$ excludes both A_n (its series starts from lower power r^{-n}) and B_n (its parity is different from n). This proves the uniqueness of $s_{(n)}$. Q.E.D.

We conclude that, restricted to the space of those $\mathbf{X}_{(n)}(f,\mathbf{Q})$ and $\mathbf{Y}_{(n)}(f,\mathbf{Q})$, for which $f \in L_{nm}$, the operator "curl_φ" has a right inverse "$\mathrm{curl}_\varphi^{-1}$" (i.e. $\mathrm{curl}_\varphi \mathrm{curl}_\varphi^{-1} = \mathrm{id}$) given by the following formulae:

$$\mathrm{curl}_\varphi^{-1}\mathbf{X}_{(n)}(f,\mathbf{Q}) = -\mathbf{Y}_{(n)}(f,\mathbf{Q}), \tag{32}$$

$$\mathrm{curl}_\varphi^{-1}\mathbf{Y}_{(n)}(f,\mathbf{Q}) = -a^{-2}\mathbf{X}_{(n)}(s_{(n)}(f),\mathbf{Q}) \ . \tag{33}$$

Let us notice that for $f \in L_{nm}$ being of minimal order $(-n+2m)$, one has that:

$$\mathbf{X}_{(n)}(f,\mathbf{Q}) \text{ is at least of order } r^{-n+2m-2},$$

$$\mathbf{Y}_{(n)}(f,\mathbf{Q}) \text{ is at least of order } r^{-n+2m-1}.$$

Those orders are called the generic orders. Observe that (generically) $\mathrm{curl}_\varphi^{-1}$ increases the order (in r) by 1.

Now, suppose that both f and $\mathbf{Q}$ are time-dependent. It turns out that also time-depending operator # (cf. (15)–(16)) acts in a simple way on the fields $\mathbf{X}$ and $\mathbf{Y}$. Indeed, we have:

Theorem 1. For $r \neq 0$

$$\begin{aligned}\mathbf{X}_{(n)}(f,\mathbf{Q})^\# = &-\frac{1}{2}\mathbf{X}_{(n+1)}(r^2 f_{,r} + (n+1)rf, \mathbf{b}\vee\mathbf{Q}) + \mathbf{Y}_{(n)}(g_1, \mathbf{b}\times\mathbf{Q}) + \\ &+\frac{n^2-1}{2n(2n+1)}\mathbf{X}_{(n-1)}(-r^2 f_{,r} + nrf, \mathbf{b}\rfloor\mathbf{Q}) + \mathbf{X}_{(n)}(\dot f,\mathbf{Q}) + \mathbf{X}_{(n)}(f,\dot{\mathbf{Q}}),\end{aligned} \tag{34}$$

$$\begin{aligned}\mathbf{Y}_{(n)}(f,\mathbf{Q})^\# = &\ \mathbf{Y}_{(n+1)}(g_2, \mathbf{b}\vee\mathbf{Q}) - \frac{1}{2(n+1)}\mathbf{X}_{(n)}(\frac{fr^2}{\varphi}, \mathbf{b}\times\mathbf{Q}) + \\ &+\mathbf{Y}_{(n-1)}(g_3, \mathbf{b}\rfloor\mathbf{Q}) + \mathbf{Y}_{(n)}(\dot f,\mathbf{Q}) + \mathbf{Y}_{(n)}(f,\dot{\mathbf{Q}}) + \mathbf{Y}_{(n)}(\frac{fr^2}{2\varphi}, a\dot a\mathbf{Q}),\end{aligned} \tag{35}$$

where $b_k = (a_k)^{\dot{}}$, $\dot a = da/d\tau = a^k b_k/a$, whereas the symmetric traceless tensors: $\mathbf{b}\vee\mathbf{Q}$, $\mathbf{b}\times\mathbf{Q}$ and $\mathbf{b}\rfloor\mathbf{Q}$ of ranks (respectively) $n+1$, n and $n-1$, are built from the n-tensor $\mathbf{Q}$ and the vector $\mathbf{Q}$ according to the following formulae (the letters placed below dots denote missing indices):

$$\begin{aligned}(\mathbf{b}\vee\mathbf{Q})_{i_0\cdots i_n} &= \frac{1}{n+1}\sum_{s=0}^{n} b_{i_s} Q_{i_0 \underbrace{\cdots}_{\hat s} i_n} \\ &- \frac{2}{(n+1)(2n+1)}\sum_{k<l}\delta_{i_k i_l} b^m Q_{m i_0 \underbrace{\cdots}_{\hat k \hat l} i_n},\end{aligned} \tag{36}$$

$$(\mathbf{b}\times\mathbf{Q})_{i_1\cdots i_n} = \frac{1}{n}\sum_{k=1}^{n} \epsilon_{i_k}{}^{lm} b_l Q_{m i_1 \underbrace{\cdots}_{\hat k} i_n}, \tag{37}$$

$$
\begin{aligned}
(\mathbf{b}\rfloor\mathbf{Q})_{i_2\cdots i_n} &= b^m Q_{mi_2\cdots i_n}, \qquad (38)\\
g_1 &= \frac{\varphi}{2(n+1)}[r^2 f_{,rr} + 2rf_{,r} - n(n+1)f],\\
g_2 &= -\frac{1}{2}(n+3)rf - \frac{1}{2}r^2 f_{,r} - \frac{a^2 f r^3}{4\varphi},\\
g_3 &= \frac{n^2-1}{n(4n+2)}\left[(n-2)rf - r^2 f_{,r} - \frac{a^2 f r^3}{2\varphi}\right],\\
\varphi &= 1-(ar/2)^2.
\end{aligned}
$$

Remark. For the properties of $\mathbf{b}\vee\mathbf{Q}$, $\mathbf{b}\times\mathbf{Q}$, $\mathbf{b}\rfloor\mathbf{Q}$ (computed in a simpler situation, but valid universally) see [9], pp. 304-305 (Warning: Normalization of the tensors $\mathbf{b}\vee\mathbf{Q}$ and $\mathbf{b}\times\mathbf{Q}$ used in (36)–(37) differs slightly from that used in (23)-(24) of [9]). Let us notice that for $\mathbf{W}$ being $\mathbf{X}$ or $\mathbf{Y}$ field, $\mathbf{W}^\#$ is at least of the same generic order of r as $\mathbf{W}$. If, moreover, $f, \mathbf{Q}$ do not depend upon τ, the generic order of $\mathbf{W}^\#$ increases by at least 1 with respect to the generic order of $\mathbf{W}$.

We shall use these formulae in a special case, when f depends upon time only *via* the acceleration $a = a(\tau)$ contained in the combination $u = ar/2$. Assuming, therefore, that $f(r,\tau) = f(u)$, one can rewrite (34)–(35) as follows:

$$
\begin{aligned}
\mathbf{X}_{(n)}(f,\mathbf{Q})^\# = &-\mathbf{X}_{(n+1)}\left[u^2 f_{,u} + (n+1)uf, \frac{1}{a}\mathbf{b}\vee\mathbf{Q}\right] +\\
&\frac{1}{2(n+1)}\mathbf{Y}_{(n)}[\varphi(u^2 f_{,uu} + 2uf_{,u} - n(n+1)f), \mathbf{b}\times\mathbf{Q}] +\\
&\frac{n^2-1}{n(2n+1)}\mathbf{X}_{(n-1)}\left[-u^2 f_{,u} + nuf, \frac{1}{a}\mathbf{b}\rfloor\mathbf{Q}\right] + \mathbf{X}_{(n)}[uf_{,u}, \frac{\dot a}{a}\mathbf{Q}] + \mathbf{X}_{(n)}[f, \dot{\mathbf{Q}}], \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Y}_{(n)}(f,\mathbf{Q})^\# = &-\mathbf{Y}_{(n+1)}\left[(n+3)uf + u^2 f_{,u} + \frac{2u^3 f}{\varphi}, \frac{1}{a}\mathbf{b}\vee\mathbf{Q}\right] +\\
&-\frac{2}{n+1}\mathbf{X}_{(n)}(\frac{fu^2}{\varphi}, \frac{1}{a^2}\mathbf{b}\times\mathbf{Q}) +\\
&\frac{n^2-1}{n(2n+1)}\mathbf{Y}_{(n-1)}\left[(n-2)uf - u^2 f_{,u} - \frac{2u^3 f}{\varphi}, \frac{1}{a}\mathbf{b}\rfloor\mathbf{Q}\right] +\\
&\mathbf{Y}_{(n)}[uf_{,u}, \frac{\dot a}{a}\mathbf{Q}] + \mathbf{Y}_{(n)}[f, \dot{\mathbf{Q}}] + \mathbf{Y}_{(n)}[\frac{fu^2}{\varphi}, \frac{2\dot a}{a}\mathbf{Q}]. \qquad (40)
\end{aligned}
$$

The proof of Theorem 1 is given in Appendix A.

5 Expansion of the Maxwell field

In this section we find an asymptotic solution for the Maxwell field in a vicinity of a time-dependent, electric 2^n − pole, moving along a given trajectory.

Given a timelike trajectory, we construct its modified Fermi frame like in Section 2 and consider Maxwell equations (17)-(20) with electric charge:

$$\mathcal{J}^0 = c_n Q^{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} \delta^{(3)} \tag{41}$$

and the corresponding electric current

$$\mathcal{J}^k = -c_n \dot{Q}^{k i_2 \cdots i_n} \partial_{i_2} \dots \partial_{i_n} \delta^{(3)}, \tag{42}$$

at the right-hand side. The above four-current is conserved: $\partial_\mu \mathcal{J}^\mu = 0$, cf. [9]. Here, $c_n = (-1)^n 4\pi/(2n-1)!!$ is the normalization factor. For $n = 0$ we assume $\mathcal{J}^k = 0$ and $Q = \text{const}$. (See Appendix C for relation of these 2^n–poles, defined in the modified Fermi frame, with the standard 2^n-poles in the Fermi frame.)

Theorem 2. Maxwell equations (17)–(20) with the sources (41)-(42) are solved by the following (formal) series:

$$\mathcal{D} = \mathcal{D}_d + \mathcal{D}_{(-n-2)} + \mathcal{D}_{(-n)} + \mathcal{D}_{(-n+2)} + \dots, \tag{43}$$

$$\mathcal{B} = \mathcal{B}_{(-n-1)} + \mathcal{B}_{(-n+1)} + \mathcal{B}_{(-n+3)} + \dots, \tag{44}$$

where

$$\mathcal{D}_d^i = (-1)^n \frac{4\pi n}{(2n+1)!!} Q^{i i_2 \dots i_n} \partial_{i_2} \dots \partial_{i_n} \delta^{(3)}, \tag{45}$$

$$\mathcal{D}_{(-n-2)} = \mathbf{X}_{(n)}(k_n P_n(z), \mathbf{Q}), \tag{46}$$

$$\mathcal{B}_{(k+1)} = \text{curl}_\varphi^{-1}[\mathcal{D}_{(k)}{}^\#], \quad k = -n-2, -n, -n+2, \dots, \tag{47}$$

$$\mathcal{D}_{(k+1)} = -\text{curl}_\varphi^{-1}[\mathcal{B}_{(k)}{}^\#], \quad k = -n-1, -n+1, -n+3, \dots, \tag{48}$$

i.e. $\mathcal{B}_{(-n-1)} = \text{curl}_\varphi^{-1}[\mathcal{D}_{(-n-2)}{}^\#]$, $\mathcal{D}_{(-n)} = -\text{curl}_\varphi^{-1}[\mathcal{B}_{(-n-1)}{}^\#]$, $\mathcal{B}_{(-n+1)} = \text{curl}_\varphi^{-1}[\mathcal{D}_{(-n)}{}^\#]$, ..., $k_n = (2a)^n/\binom{2n}{n}$. Moreover, $\mathcal{B}_{(l)}$ and $\mathcal{D}_{(l)}$ are at least of order r^l.

Remark 1. The fields $\mathcal{B}, \mathcal{D}$ and the operations $\text{div}, \text{curl}_\varphi, \#$ are considered here in the sense of distributions. On the other hand, curl_φ^{-1} acts on

the fields $\mathbf{X}$ and $\mathbf{Y}$ via (32)–(33). One can add to (17) – (20) any regular solution $(\mathcal{D},\mathcal{B})$ of homogeneous Maxwell equations.

Remark 2. One has $k_n P_n(z) = r^{-n} + f_n$, where $\lim_{r\to 0} f_n(r)r^n = 0$. Thus the leading term in $\mathcal{D}$ equals $\mathcal{D}_d + \mathbf{X}_{(n)}(r^{-n},\mathbf{Q}) = -\mathrm{grad}\psi_n$ where

$$\psi_n = \frac{Q^{i_1\ldots i_n} x_{i_1}\ldots x_{i_n}}{r^{2n+1}}$$

is the standard potential for the classical 2^n-pole (41) with $a\equiv 0$, $\mathbf{Q} = \mathrm{const.}$ (cf. Appendix D).

Proof of Theorem 2 is given in Appendix D.

6 Application to the monopole and the dipole cases

In this section we apply the above formulae to the case of a monopole and a dipole particle. We find explicitly asymptotic expansion of the Maxwell field $(\mathcal{D},\mathcal{B})$ up to the order r^1.

In the monopole case we start from $\mathcal{D}_{(-2)} = \mathbf{X}_{(0)}(1,Q)$ which is of order r^{-2}, in the dipole case we start from $\mathcal{D}_{(-3)} = \mathbf{X}_{(1)}(k_1P_1(z),\mathbf{Q}) = \mathbf{X}_{(1)}(\frac{1}{u},\frac{a}{2}\mathbf{Q}) + \mathbf{X}_{(1)}(u,\frac{a}{2}\mathbf{Q})$. The last two components are, respectively, of orders r^{-3} and r^{-1}. To compute the next terms we use operations $\mathrm{curl}_\varphi^{-1}$ and #. While $\mathrm{curl}_\varphi^{-1}$ generically increases the order by one, (39)–(40) show that # applied to $\mathbf{X}_{(n)}(f,Q)$ or $\mathbf{Y}_{(n)}(f,Q)$ leads to terms of different orders: in generic case, first three terms of (39) or (40) have order increased by one, the next two terms there have order unchanged and the last term in (40) has order increased by two. We present the results of computation in the monopole and in the dipole cases in terms of $u = ar/2$ and $R = \log(\frac{1-u}{1+u}) = -2u + \ldots$, obtained using the symbolic calculus provided by the program MAPLE 8. The terms of order higher than r (or u) are denoted by $o(u)$. We collect all the terms of generic order r^l from different parts of (43) or (44) and denote them, respectively, by $\mathcal{D}^{(l)}$ or $\mathcal{B}^{(l)}$. In the monopole case we assume that the charge $Q = \mathrm{const.}$

The monopole case

$$\mathcal{D} = \mathcal{D}^{(-2)} + \mathcal{D}^{(1)} + o(u), \quad \mathcal{B} = \mathcal{B}^{(0)} + o(u), \tag{49}$$

where

$$\mathcal{D}^{(-2)} = \mathbf{X}_{(0)}(1, Q), \quad \mathcal{D}^{(1)} = \mathbf{X}_{(1)}(f_0, \frac{1}{a^3}Q\dot{\mathbf{b}}), \quad \mathcal{B}^{(0)} = \mathbf{Y}_{(1)}(u, \frac{1}{a}Q\mathbf{b}), \quad (50)$$

with

$$f_0 = -\frac{1}{4}(u + u^{-1})R^2 - R - u = -u^3 + \ldots. \quad (51)$$

The dipole case

$$\mathcal{D} = -\frac{4\pi}{3}\mathbf{Q}\delta^{(3)} + \mathcal{D}^{(-3)} + \mathcal{D}^{(-1)} + \mathcal{D}^{(0)} + \mathcal{D}^{(1)} + o(u), \quad (52)$$

$$\mathcal{B} = \mathcal{B}^{(-2)} + \mathcal{B}^{(-1)} + \mathcal{B}^{(0)} + \mathcal{B}^{(1)} + o(u), \quad (53)$$

where

$$\mathcal{D}^{(-3)} = \mathbf{X}_{(1)}(u^{-1}, \frac{a}{2}\mathbf{Q}), \quad (54)$$

$$\mathcal{D}^{(-1)} = -\mathbf{X}_{(1)}(f_1, \frac{1}{2a}\ddot{\mathbf{Q}}) + \mathbf{X}_{(1)}(u, \frac{a}{2}\mathbf{Q}), \quad (55)$$

$$\begin{aligned}\mathcal{D}^{(0)} = {} & \mathbf{X}_{(2)}(f_2, \frac{1}{2a^2}(\mathbf{b} \vee \mathbf{Q})^{\cdot}) + \mathbf{Y}_{(1)}(u, \frac{1}{2a}(\mathbf{b} \times \mathbf{Q})^{\cdot}) + \\ & + \mathbf{Y}_{(1)}(\frac{u}{1-u^2}, \frac{1}{2a}\mathbf{b} \times \dot{\mathbf{Q}}) + \mathbf{X}_{(2)}(f_3, \frac{1}{2a^2}\mathbf{b} \vee \dot{\mathbf{Q}}), \end{aligned} \quad (56)$$

$$\begin{aligned}\mathcal{D}^{(1)} = {} & -\mathbf{Y}_{(2)}(\frac{u^2}{1-u^2}, \frac{1}{3a^2}\mathbf{b} \times (\mathbf{b} \vee \mathbf{Q})) - \mathbf{X}_{(3)}(f_4, \frac{1}{2a^3}\mathbf{b} \vee (\mathbf{b} \vee \mathbf{Q})) + \\ & -\mathbf{Y}_{(2)}(u^2, \frac{3}{2a^2}\mathbf{b} \vee (\mathbf{b} \times \mathbf{Q})) - \mathbf{X}_{(1)}(f_5, \frac{\dot{a}}{a^2}\dot{\mathbf{Q}}) + \\ & -\mathbf{X}_{(1)}(f_0, \frac{\dot{a}}{a^3}(\dot{a}\mathbf{Q} + \frac{a}{2}\dot{\mathbf{Q}})) - \mathbf{X}_{(1)}(f_0, \frac{1}{a^2}(\dot{a}\mathbf{Q} + \frac{a}{2}\dot{\mathbf{Q}})^{\cdot}) + \mathbf{X}_{(1)}(f_8, \frac{\dot{a}^2}{2a^5}\ddot{\mathbf{Q}}) + \\ & \mathbf{X}_{(1)}\left[f_7, \frac{\dot{a}}{a^3}(\frac{\ddot{\mathbf{Q}}}{2a})^{\cdot} + \frac{1}{a^2}(\frac{\dot{a}\ddot{\mathbf{Q}}}{2a^2})^{\cdot}\right] + \mathbf{X}_{(1)}(f_6, \frac{1}{a^2}(\frac{\ddot{\mathbf{Q}}}{2a})^{\cdot\cdot}), \end{aligned} \quad (57)$$

$$\mathcal{B}^{(-2)} = -\mathbf{Y}_{(1)}(u^{-1}, \frac{a}{2}\dot{\mathbf{Q}}), \quad (58)$$

$$\mathcal{B}^{(-1)} = \frac{1}{2}\mathbf{Y}_{(2)}(1, \mathbf{b} \vee \mathbf{Q}) + \mathbf{X}_{(1)}(u, \frac{1}{2a}\mathbf{b} \times \mathbf{Q}), \quad (59)$$

$$\mathcal{B}^{(0)} = -\mathbf{Y}_{(1)}(u, \dot{a}\mathbf{Q} + \frac{a}{2}\dot{\mathbf{Q}}) + \mathbf{Y}_{(1)}(f_1, (\frac{\ddot{\mathbf{Q}}}{2a})^{\cdot}\,) + \mathbf{Y}_{(1)}(f_9, \frac{\dot{a}}{2a^2}\ddot{\mathbf{Q}}), \tag{60}$$

$$\begin{aligned}
\mathcal{B}^{(1)} = \frac{3}{2}\mathbf{Y}_{(2)}(u^2, \mathbf{b}\vee\mathbf{Q}) - \mathbf{Y}_{(2)}(f_{10}, \frac{\dot{a}}{2a^3}(\mathbf{b}\vee\mathbf{Q})^{\cdot}\,) - \\
\mathbf{Y}_{(2)}(f_2, [\frac{1}{2a^2}(\mathbf{b}\vee\mathbf{Q})^{\cdot}\,]^{\cdot}\,) - \mathbf{X}_{(1)}(f_0, \frac{\dot{a}}{2a^4}(\mathbf{b}\times\mathbf{Q})^{\cdot}\,) - \\
\mathbf{X}_{(1)}(f_0, \frac{1}{a^2}[\frac{1}{2a}(\mathbf{b}\times\mathbf{Q})^{\cdot}\,]^{\cdot}\,) - \mathbf{Y}_{(2)}(f_{11}, \frac{\dot{a}}{2a^3}\mathbf{b}\vee\dot{\mathbf{Q}}) - \\
\mathbf{Y}_{(2)}(f_3, (\frac{1}{2a^2}\mathbf{b}\vee\dot{\mathbf{Q}})^{\cdot}\,) - \mathbf{X}_{(1)}(f_{12}, \frac{\dot{a}}{2a^4}\mathbf{b}\times\dot{\mathbf{Q}}) - \\
\mathbf{X}_{(1)}(f_5, \frac{1}{a^2}(\frac{1}{2a}\mathbf{b}\times\dot{\mathbf{Q}})^{\cdot}\,) - \mathbf{Y}_{(2)}(f_{13}, \frac{1}{2a^2}\mathbf{b}\vee\ddot{\mathbf{Q}}) + \\
\mathbf{X}_{(1)}(f_{14}, \frac{1}{8a^3}\mathbf{b}\times\ddot{\mathbf{Q}}),
\end{aligned} \tag{61}$$

where f_0 is given by (51),

$$f_1 = -\frac{1}{4}(u^{-1}+u)R^2 - R + u = 2u + \ldots,$$

$$f_2 = \left[-\frac{3}{16}(u^{-1}+u)^2 + \frac{1}{4}\right]R^2 - \frac{3}{4}(u^{-1}+u)R - \frac{3}{4} = u^2 + \ldots,$$

$$f_3 = -\frac{1}{8}(3u^{-2}+2+3u^2)R^2 - \frac{3}{2}(u^{-1}+u)R - \frac{3-5u^2+6u^4}{2(1-u^2)} = 3u^2 + \ldots,$$

$$\begin{aligned}
f_4 &= -\frac{3}{32}(5u^{-3}+3u^{-1}+3u+5u^3)R^2 - \frac{1}{8}(15u^{-2}+14+15u^2)R \\
&- \frac{15u^{-1}+4u-23u^3+20u^5}{8(1-u^2)} = \frac{10}{3}u^3 + \ldots,
\end{aligned}$$

$$f_5 = -\frac{u^3}{1-u^2} = -u^3 + \ldots,$$

$$f_6 = \frac{1}{48}(u^{-1}+u)R^4 + \frac{1}{6}R^3 - \frac{1}{4}(u^{-1}+u)R^2 - R - u = -2u^3 + \ldots,$$

$$f_7 = -\frac{1}{2u}R^2 - \frac{2}{1-u^2}R - \frac{2u+4u^3}{1-u^2} = -2u^3 + \ldots,$$

$$f_8 = \frac{1}{48}(u^{-1}+u)R^4 + \frac{1}{6}R^3 - \frac{3}{8}(u^{-1}+u)R^2 -$$

$$\frac{3-4u^2+3u^4}{2(1-u^2)^2}R-\frac{3u-3u^3+2u^5}{2(1-u^2)^2}=-2u^3+\ldots,$$

$$f_9=\frac{1}{4}(u^{-1}-u)R^2+\frac{1+u^2}{1-u^2}R+\frac{3u-u^3}{1-u^2}=2u+\ldots,$$

$$f_{10}=\frac{3}{8}(u^{-2}-u^2)R^2+\frac{3u^{-1}-2u+3u^3}{2(1-u^2)}R+\frac{3(1+u^2)}{2(1-u^2)}=2u^2+\ldots,$$

$$f_{11}=\frac{3}{4}(u^{-2}-u^2)R^2+\frac{3u^{-1}-2u+3u^3}{1-u^2}R+\frac{3+2u^2-15u^4+6u^6}{(1-u^2)^2}=6u^2+\ldots,$$

$$f_{12}=-\frac{1}{16}(u^{-1}+u)R^2-\frac{1}{4}R-\frac{u+u^3}{4(1-u^2)^2}=-u^3+\ldots,$$

$$f_{13}=-\frac{1}{4}(1+3u^2)R^2-\frac{u-3u^3}{1-u^2}R+\frac{5u^2-3u^4}{1-u^2}=6u^2+\ldots,$$

$$f_{14}=-\frac{1}{u}R^2-\frac{4}{1-u^2}R-\frac{4(u+u^3)}{1-u^2}=\frac{20}{9}u^5+\ldots.$$

7 Description of the functions appearing in the Maxwell fields (a conjecture)

The method of analyzing Maxwell equations in a neighbourhood of a freely moving electric multipole leads, in case of a monopole and a dipole particle, to an interesting family of relatively simple functions of the variable r. These functions are obtained from $A_n(r)=P_n(z)$ contained in (46) by consecutive application of the differential operators contained in # and the integral operator contained in $\mathrm{curl}_\varphi^{-1}$. The latter is based on solving the differential equation (30) and could *a priori* lead very quickly to non-elementary functions. An unexpected result of our analysis consists in the fact, that the functions arising here are polynomials of the universal quantity $R=\log(\frac{1-u}{1+u})=\log(\frac{2-ar}{2+ar})$, with coefficients being rational combinations of $u=ar/2$ itself. This fact leads us to a conjecture, that a new family of special functions arises here, which makes the physical picture of the field relatively simple.

In this Section we present a conjecture about the smallest algebra of functions f appearing in $\mathbf{X}_{(k)}(f,\mathbf{Q})$, $\mathbf{Y}_{(k)}(f,\mathbf{Q})$, when we solve Maxwell equations with multipole particles, using the method proposed in this paper.

We set $R_s = \frac{R^s}{2^s s!}$. First we notice that only constant functions appear in $\mathbf{X}_{(0)}$ (remember that $\mathbf{Y}_{(0)} = 0$). Indeed, the leading term for the monopole particle contains only $P_0(z) = 1$ and the terms with $\mathbf{X}_{(0)}$ produced by (39) – (40) vanish.

We define H_k as the linear span of functions f entering $\mathbf{X}_{(k)}(f, \mathbf{Q})$, $\mathbf{Y}_{(k)}(f, \mathbf{Q})$, which appear when we consecutively use formulae (32) – (33) or (39) – (40), starting from multipole fields (46). More precisely, we take the following

Definition. We set H_k, $k = 1, 2, \ldots$, as the smallest vector spaces such that

$$u^{-k}, u^{-k+2}, \ldots u^k \in H_k, \tag{62}$$

and such that for any $f \in H_k$ one has $s_{(k)}(f) \in H_k$,

$$\frac{u^2 f}{1-u^2} \in H_k,\ u f_{,u} \in H_k,\ (1-u^2)[u^2 f_{,uu} + 2u f_{,u} - k(k+1)f] \in H_k, \tag{63}$$

$$u^2 f_{,u} + (k+1)uf \in H_{k+1},\ u^2 f_{,u} + (k+3)uf + \frac{2u^3 f}{1-u^2} \in H_{k+1}, \tag{64}$$

$$-u^2 f_{,u} + kuf \in H_{k-1},\ u^2 f_{,u} - (k-2)uf + \frac{2u^3 f}{1-u^2} \in H_{k-1} \quad (k \neq 1). \tag{65}$$

Remark. Due to condition (62) we have:

$$P_n(z) = P_n(\frac{1}{2}(u^{-1} + u)) \in \mathrm{span}\{u^{-n}, u^{-n+2}, \ldots, u^n\}\ ,$$

and, whence, functions f contained in $\mathcal{D}_{(-n-2)} = \mathbf{X}_{(n)}(k_n P_n(z), \mathbf{Q})$ are in H_n. Also for $n = 0$ and f appearing in $\mathcal{B}_{(-1)} = \mathbf{Y}_{(1)}(u, \frac{1}{a} Q\mathbf{b})$ we have $f = u \in H_1$. It remains to consider operations $\mathrm{curl}_{\varphi}^{-1}$ and # which yield $\mathcal{B}_{(l)}$ and $\mathcal{D}_{(l)}$ with higher l. Due to (32)–(33) and (39)–(40), conditions (63)–(65) imply that also f appearing in those $\mathcal{B}_{(l)}$, $\mathcal{D}_{(l)}$ belong to the corresponding H_k.

It is easy to see that conditions (63)–(65) can be reformulated in a simpler, equivalent form:

$$\frac{f}{1-u^2} \in H_k,\ u f_{,u} \in H_k,\ (1-u^2)[u^2 f_{,uu} + 2u f_{,u} - k(k+1)f] \in H_k, \tag{66}$$

$$\frac{uf}{1-u^2} \in H_{k+1},\ u^2 f_{,u} + (k+1)uf \in H_{k+1}, \tag{67}$$

$$\frac{uf}{1-u^2} \in H_{k-1},\ u^2 f_{,u} - kuf \in H_{k-1} \quad (k \neq 1). \tag{68}$$

Conjecture: The space H_k can be represented as

$$H_k = \bigcup_{m>k} H_{km} \quad \text{(increasing sequence of vector spaces)}, \tag{69}$$

where H_{km} are the direct sums:

$$H_{km} = \bigoplus_{j=0}^{\infty} H_{kjm}, \tag{70}$$

$$H_{k0m} = \operatorname{span}\left\{\frac{u^{-k}}{(u^2-1)^m}, \frac{u^{-k+2}}{(u^2-1)^m}, \dots, \frac{u^{k+2m}}{(u^2-1)^m}\right\}, \quad m = 0, 1, 2, \dots, \tag{71}$$

$$H_{k,2l-1,m} = \operatorname{span}\left\{\frac{R^{2l-1}u^{k+1}}{(u^2-1)^m}, \frac{R^{2l-1}u^{k+3}}{(u^2-1)^m}, \dots, \frac{R^{2l-1}u^{2m-k-1}}{(u^2-1)^m}\right\}, \quad (m > k), \tag{72}$$

$$\begin{aligned} H_{k,2l,m} = \operatorname{span}\Bigg\{ & \frac{w_k^l(R,u)}{u^k(u^2-1)^m}, \frac{w_{k-1}^l(R,u)}{u^{k-2}(u^2-1)^m}, \dots, \frac{w_1^l(R,u)}{u^{-k+2}(u^2-1)^m}, \\ & \frac{R^{2l}u^k}{(u^2-1)^m}, \frac{R^{2l}u^{k+2}}{(u^2-1)^m}, \dots, \frac{R^{2l}u^{2m-k}}{(u^2-1)^m}, \frac{w_1^l(R,u^{-1})u^{2m-k+2}}{(u^2-1)^m}, \\ & \frac{w_2^l(R,u^{-1})u^{2m-k+4}}{(u^2-1)^m}, \dots, \frac{w_k^l(R,u^{-1})u^{2m+k}}{(u^2-1)^m}\Bigg\}, \quad (m > k), \end{aligned} \tag{73}$$

$l = 1, 2, \dots$, with w_k^l given recursively by $w_0^l(R,u) = R_{2l}$,

$$w_k^l(R,u) = \left(\sum_{j=1}^{\min(k,2l-1)} A_{kj} w_{k-j}^{l-[j/2]}(R,u) u^{2j-2}\right) + (2k-3)!! R_{2l-1} u^{2k-1}, \tag{74}$$

$k, l = 1, 2, \dots$ (here, we put $(-1)!! = 1$, so that $(-1)!! \cdot 1 = 1!!$), for some real numbers A_{kj}, $k = 1, 2, \dots$, $j = 1, 2, \dots, k$. Moreover, we conjecture that $A_{k1} = 2k-1$, $A_{k2} = 1$, $A_{kk} = (2k-3)!!$. Unfortunately, we were not able to find a general formula for A_{kj}.

Remark 1. We know only a recursive procedure to find A_{kj}: Suppose we have found all A_{rj}, $r < k$. Then we can compute w_r^l, $r < k$ (cf. (74)). We set

$$f = \frac{w_{k-1}^l(R,u^{-1})u^{2m+k-2}}{(u^2-1)^m},$$

i.e. f is the last but one element of (73). Then we compute $g \in H_k$ by means of the last formula of (66). Next we decompose g into the bases of $H_{k,j,m+1}$, $j \leq 2l$, where w_k^l is given by (74). This gives us an equation which may be used to find coefficients A_{kj}, $j = 1, 2, \ldots, k$. That kind of procedure was implemented by us for $k \leq 10$, using the symbolic calculus provided by the program MAPLE 8. In particular, we have obtained the following results for $k \leq 6$:

$$A_{11} = 1, \tag{75}$$

$$A_{21} = 3,\ A_{22} = 1, \tag{76}$$

$$A_{31} = 5,\ A_{32} = 1,\ A_{33} = 3, \tag{77}$$

$$A_{41} = 7,\ A_{42} = 1,\ A_{43} = 50,\ A_{44} = 15, \tag{78}$$

$$A_{51} = 9,\ A_{52} = 1,\ A_{53} = 1273/9,\ A_{54} = 485/9,\ A_{55} = 105, \tag{79}$$

$$A_{61} = 11, A_{62} = 1, A_{63} = 62564/225, A_{64} = 2703/25, A_{65} = 9985/3, A_{66} = 945. \tag{80}$$

Using (74), one gets that

$$w_k^l(R,u) = \sum_{s=0}^{\min(k,2l-1)} g_{ks}(u) R_{2l-s}\,, \tag{81}$$

where g_{ks} are polynomials. In particular, we have:

$$w_0^l(R,u) = R_{2l}, \tag{82}$$

$$w_1^l(R,u) = R_{2l} + uR_{2l-1}, \tag{83}$$

$$w_2^l(R,u) = 3R_{2l} + (3u + u^3)R_{2l-1} + u^2 R_{2l-2}, \tag{84}$$

$$w_3^l(R,u) = 15R_{2l} + (15u + 5u^3 + 3u^5)R_{2l-1} + (6u^2 + 3u^4)R_{2l-2} + u^3 R_{2l-3}. \tag{85}$$

(Observe that, due to (81), for $l = 1$ we omit the last term in w_2^l and the last two terms in w_3^l.)

Remark 2. One can prove that for $A_{k1} = 2k - 1$ and any value of the remaining coefficients A_{kj}, the elements (71)–(73), spanning each vector space H_{kjm} are linearly independent and the direct sum condition is obvious. Moreover, using (74), H_{km} form an increasing sequence of vector spaces and therefore H_k are well defined. The elements (62) belong to $H_{k00} \subset H_{k0m} \subset H_k$ $(m > k)$. Using once again (74), it can be also checked that the first operations of (66)–(68) provide elements of H_k, H_{k+1} and H_{k-1}, respectively.

It remains to find the formula for A_{kj} such that the remaining four relations of (66)–(68) are fulfilled, to prove that $s_{(k)}(f) \in H_k$ for $f \in H_k$ and to show the minimality of H_k.

Remark 3. Elements of H_k are rational functions of u and R. However, the generators of these spaces, proposed in formulae (71)–(73) are quite complicated. But splitting the nominators into monomials doesn't simplify the situation, because application of $s_{(k)}$ (which must preserve H_k) to the components obtained in that way leads to functions which seem to be in general not rational in u and R. Thus the form (71)–(73) of the generators seems to be the simplest one.

Appendices

A Properties of fields X and Y.

We assume $r \neq 0$. Using (23)–(24), after some computations we get

$$\partial_l X_k = [-r^2 f_{,rr} + (2n+3) r f_{,r} - (n+1)(n+3) f] \frac{x_l x_k}{r^{n+5}} H +$$

$$[(n+1) f - r f_{,r}] \frac{1}{r^{n+3}} \delta_{kl} H + n[(n+1) f - r f_{,r}] \frac{x_k Q_l}{r^{n+3}} +$$

$$(r f_{,rr} - n f_{,r}) \frac{x_l Q_k}{r^{n+2}} + \frac{f_{,r}}{r^n} (n-1) Q_{kl},$$

$$\partial_l Y_k = [\frac{r f_{,r}}{\varphi} + \frac{a^2 r^2 f}{2\varphi^2} - \frac{(n+1) f}{\varphi}] \frac{x_l E_k}{r^{n+3}} +$$

$$\frac{f}{\varphi r^{n+1}} \epsilon_{kl}{}^m Q_m + \frac{(n-1) f}{\varphi r^{n+1}} F_{kl},$$

where $H = Q_{i_1 \dots i_n} x^{i_1} x^{i_2} \dots x^{i_n}$, $Q_k = Q_{k i_2 \dots i_n} x^{i_2} \dots x^{i_n}$, $Q_{kl} = Q_{k l i_3 \dots i_n} x^{i_3} \dots x^{i_n}$, $E_k = \epsilon_k{}^{jm} x_j Q_m$, $F_{kl} = \epsilon_k{}^{jm} x_j Q_{ml}$. That implies (25), (26)–(27),

$$x_l X^l = \frac{(n+1) f}{r^{n+1}} H, \quad x_l Y^l = 0 \tag{86}$$

and (cf. (16))

$$\tilde{X}_k = \frac{1}{2} [r^2 f_{,rr} + r f_{,r} - (n+1)^2 f] \frac{x_k (b^l x_l) H}{r^{n+3}} +$$

$$-\frac{1}{2}[rf_{,r}+(n+1)f]\frac{b_kH}{r^{n+1}}-\frac{1}{2}[(n-2)rf_{,r}-n(n+1)f]\frac{(b^mQ_m)x^k}{r^{n+1}}+$$
$$-\frac{1}{2}[rf_{,rr}+(n+2)f_{,r}]\frac{(b^lx_l)Q_k}{r^n}+\frac{n-1}{2}\frac{f_{,r}}{r^{n-2}}b^lQ_{kl}, \quad (87)$$

$$\tilde{Y}_k=-\frac{1}{2}[\frac{rf_{,r}}{\varphi}+\frac{a^2r^2f}{2\varphi^2}+\frac{(n+3)f}{\varphi}]\frac{E_k(b^lx_l)}{r^{n+1}}+$$
$$\frac{f}{2\varphi r^{n-1}}G_k+\frac{(n-1)f}{2\varphi r^{n-1}}b^lF_{kl}+\frac{fx_k}{\varphi r^{n+1}}(b^lE_l), \quad (88)$$

where $G_k=\epsilon_k{}^{lm}b_lQ_m$. Let us notice that $Q^m{}_m=0$ ($\mathbf{Q}$ is traceless) implies

$$0=2\delta_{ms}Q^{ms}\epsilon^{al}{}_kx_ab_l=D_{ijs}\epsilon^{ijs}\epsilon^{al}{}_kx_ab_l,$$

where $D_{ijs}=\epsilon_{ij}{}^mQ_{ms}$. That and

$$\epsilon_{ijs}\epsilon_{alk}=\delta_{ia}(\delta_{jl}\delta_{sk}-\delta_{jk}\delta_{sl})+\delta_{il}(\delta_{jk}\delta_{sa}-\delta_{ja}\delta_{sk})+\delta_{ik}(\delta_{ja}\delta_{sl}-\delta_{jl}\delta_{sa})$$

give

$$\epsilon^{alm}x_ab_lQ_{mk}=-b^lF_{kl}+G_k. \quad (89)$$

Using (36)–(38) and (89), we obtain

$$X_{k(n+1)}(\alpha,\mathbf{b}\vee\mathbf{Q})=[(n+2)\alpha-r\alpha_{,r}]\frac{x_k}{r^{n+4}}[(b^lx_l)H-\frac{n}{2n+1}r^2(b^mQ_m)]+$$
$$\frac{\alpha_{,r}}{r^{n+1}}\{\frac{1}{n+1}[b_kH+nQ_k(b^lx_l)]-$$
$$\frac{2}{(n+1)(2n+1)}[nx_k(b^mQ_m)+\frac{n(n-1)}{2}r^2b^mQ_{mk}]\}, \quad (90)$$

$$Y_{k(n)}(\beta,\mathbf{b}\times\mathbf{Q})=\frac{\beta}{n\varphi r^{n+1}}[nb_kH-(n-1)x_k(Q^mb_m)-n(b^lx_l)Q_k+(n-1)r^2b^lQ_{kl}], \quad (91)$$

$$X_{k(n-1)}(\gamma,\mathbf{b}\rfloor\mathbf{Q})=(n\gamma-r\gamma_{,r})\frac{x_k(b^mQ_m)}{r^{n+2}}+\frac{\gamma_{,r}}{r^{n-1}}b^mQ_{mk} \quad (n\geq 2), \quad (92)$$

$$Y_{k(n+1)}(\psi,\mathbf{b}\vee\mathbf{Q})=\frac{\psi}{\varphi r^{n+2}}[\frac{1}{n+1}(\epsilon_k{}^{jl}x_jb_l)H+\frac{n}{n+1}E_k(b^lx_l)-\frac{n(n-1)}{(n+1)(2n+1)}r^2b^lF_{kl}], \quad (93)$$

$$X_{k(n)}(\kappa,\mathbf{b}\times\mathbf{Q})=\frac{\kappa_{,r}}{r^n}[G_k-\frac{n-1}{n}b^lF_{kl}]-[(n+1)\kappa-r\kappa_{,r}]\frac{x_k(b^lE_l)}{r^{n+3}}, \quad (94)$$

$$Y_{k(n-1)}(\rho, \mathbf{b}\rfloor\mathbf{Q}) = \frac{\rho}{\varphi r^n} b^l F_{kl} \quad (n \geq 2). \tag{95}$$

Computing $\epsilon_l{}^{km}\epsilon_m{}^{ij}\epsilon_j{}^{ac}x_k x_i Q_a b_c$ by means of $\epsilon_{lk}{}^m \epsilon_{mij} = \delta_{li}\delta_{kj} - \delta_{lj}\delta_{ki}$ or $\epsilon_{mi}{}^j \epsilon_{jac} = \delta_{ma}\delta_{ic} - \delta_{mc}\delta_{ia}$, one obtains

$$(\epsilon_k{}^{jl} x_j b_l)H = (b^l x_l)E_k - x_k(b^l E_l) - r^2 G_k. \tag{96}$$

Proof of Theorem 1. Comparing (87)–(88) with (90)–(95) and using (96), we get

$$\tilde{X}_k = X_{k(n+1)}(\alpha, \mathbf{b} \vee \mathbf{Q}) + Y_{k(n)}(\beta, \mathbf{b} \times \mathbf{Q}) + X_{k(n-1)}(\gamma, \mathbf{b}\rfloor\mathbf{Q}), \tag{97}$$

$$\tilde{Y}_k = Y_{k(n+1)}(\psi, \mathbf{b} \vee \mathbf{Q}) + X_{k(n)}(\kappa, \mathbf{b} \times \mathbf{Q}) + Y_{k(n-1)}(\rho, \mathbf{b}\rfloor\mathbf{Q}), \tag{98}$$

for $\alpha = -\frac{1}{2}[r^2 f_{,r} + (n+1)rf]$, $\beta = g_1$, $\gamma = \frac{n^2-1}{2n(2n+1)}(-r^2 f_{,r} + nrf)$, $\psi = g_2$, $\kappa = -\frac{1}{2(n+1)}\frac{fr^2}{\varphi}$, $\rho = g_3$.

Using (23)–(24), we obtain that $\dot{X}_k$ gives the last two terms in (34) while $\dot{Y}_k$ gives the last three terms in (35). That, (15) and (97)–(98) prove (34)–(35). Q.E.D.

B Proof of the properties of A_n and B_n

The quantity

$$z^k = \left[\frac{1}{2}\left(\frac{2}{ar} + \frac{r}{2a}\right)\right]^k = 2^{k-1}a^{-k}r^{-k} + \ldots$$

is of order r^{-k}. Therefore $P_n(z)$ (with the highest term of order z^n) is a Laurent polynomial of order r^{-n} and analogously v_{n-1} is of order r^{-n+1}. Moreover,

$$\frac{1}{2}\log\frac{z+1}{z-1} = \log\frac{1+u}{1-u} = ar + \ldots \tag{99}$$

and, therefore, $B_n(r)$ can be written as a Laurent series in r, starting from (at least) r^{-n+1}. Inserting this series into $h_{(n)}(B) = 0$, using (28) and denoting the order of $B_n(r)$ by $l \geq -n+1$, we get that vanishing of the r^{l-2} term in $h_{(n)}(B)$ implies $l = -n$ (which is impossible) or $l = n+1$. Hence, $B_n(r)$ is of order r^{n+1}.

C The transformation of 2^n–poles ($n = 0, 1, 2$)

Let $\mathcal{I}^\lambda$ be the current density in the Fermi frame (coordinates ξ) and $\mathcal{J}^\mu$ be the corresponding current density in the modified Fermi frame (coordinates x). Then

$$\mathcal{J}^\mu = \frac{\partial x^\mu}{\partial \xi^\lambda} \left| \det \left(\frac{\partial \xi}{\partial x} \right) \right| \mathcal{I}^\lambda.$$

Integrating with the test function $f(x)$, we obtain

$$S^\mu \equiv \int \mathcal{J}^\mu(x) f(x) d^3x = \int \frac{\partial x^\mu}{\partial \xi^\lambda} \mathcal{I}^\lambda |\det(\frac{\partial \xi}{\partial x})| f(x) d^3x = \int \frac{\partial x^\mu}{\partial \xi^\lambda} \mathcal{I}^\lambda(\xi) \tilde{f}(\xi) d^3\xi, \tag{100}$$

where $\tilde{f}(\xi) = f(x(\xi))$. Differentiating (3), one obtains

$$\frac{\partial x^k}{\partial \xi^l} = (\delta^k_l + a^k \xi_l) M - (\xi^k + \frac{1}{2} a^k \rho^2)(a_l + \frac{1}{2} a^2 \xi_l) M^2, \tag{101}$$

$$\frac{\partial x^k}{\partial \xi^0} = \frac{1}{2} \dot{a}^k \rho^2 M - (\xi^k + \frac{1}{2} a^k \rho^2)(\dot{a}_m \xi^m + \frac{1}{2} \dot{a}_m a^m \rho^2) M^2, \tag{102}$$

where $M = (1 + a_k \xi^k + \frac{1}{4} a^2 \rho^2)^{-1}$. Moreover, $\frac{\partial x^0}{\partial \xi^l} = 0$, $\frac{\partial x^0}{\partial \xi^0} = 1$. Thus

$$S^0 = \int \mathcal{I}^0(\xi) \tilde{f}(\xi) d^3\xi, \tag{103}$$

$$S^k = \int L^k(\xi) \tilde{f}(\xi) d^3\xi, \tag{104}$$

where

$$L^k = \frac{\partial x^k}{\partial \xi^l} \mathcal{I}^l + \frac{\partial x^k}{\partial \xi^0} \mathcal{I}^0. \tag{105}$$

Suppose that $\mathcal{I}^\lambda = \mathcal{I}^\lambda_{(n)}(\xi, \mathbf{Q})$ is the standard 2^n-pole, given by the right hand sides of the formulae (41)–(42) with respect to the coordinates ξ^μ. We shall calculate the corresponding value of $\mathcal{J}^\mu = \mathcal{J}^\mu_{(n)}$ by means of (100) in the case of a monopole, dipole and quadrupole, i. e. for $n = 0, 1, 2$ (for $n = 0$ we assume $Q = \text{const.}$).

For $n = 0$ $\mathcal{I}^0_{(0)}(\xi, Q) = 4\pi Q \delta^{(3)}$, $\mathcal{I}^k_{(0)}(\xi, Q) = 0$, hence $S^0 = 4\pi Q \tilde{f}(0) = 4\pi Q f(0)$, $L^k = 0$, $S^k = 0$,

$$\mathcal{J}^\mu_{(0)} = \mathcal{I}^\mu_{(0)}(x, Q), \tag{106}$$

where $\mathcal{I}^\lambda_{(n)}(x,\mathbf{Q})$ is given by (41)–(42) with respect to the coordinates x^μ.

For $n=1$ $\mathcal{I}^0_{(1)}(\xi,\mathbf{Q}) = -4\pi Q^k\partial_k\delta^{(3)}$, $\mathcal{I}^k_{(1)}(\xi,\mathbf{Q}) = 4\pi\dot{Q}^k\delta^{(3)}$, hence $S^0 = 4\pi Q^k(\partial_k\tilde{f})(0) = 4\pi Q^k(\partial_k f)(0)$, $L^k = 4\pi\dot{Q}^k\delta^{(3)}$, $S^k = 4\pi\dot{Q}^k\tilde{f}(0) = 4\pi\dot{Q}^k f(0)$,

$$\mathcal{J}^\mu_{(1)} = \mathcal{I}^\mu_{(1)}(x,\mathbf{Q}). \tag{107}$$

For $n=2$ $\mathcal{I}^0_{(2)}(\xi,\mathbf{Q}) = \frac{4\pi}{3}Q^{kl}\partial_k\partial_l\delta^{(3)}$, $\mathcal{I}^k_{(2)}(\xi,\mathbf{Q}) = -\frac{4\pi}{3}\dot{Q}^{kl}\partial_l\delta^{(3)}$, hence $S^0 = \frac{4\pi}{3}Q^{kl}(\partial_k\partial_l\tilde{f})(0) = \frac{4\pi}{3}Q^{kl}(\partial_k\partial_l f)(0) - \frac{8\pi}{3}Q^{kl}a_l(\partial_k f)(0)$, $L^k = -\frac{8\pi}{3}Q^{km}\dot{a}_m\delta^{(3)} - \frac{8\pi}{3}\dot{Q}^{km}a_m\delta^{(3)} - \frac{4\pi}{3}\dot{Q}^{kl}\partial_l\delta^{(3)}$, $S^k = -\frac{8\pi}{3}(Q^{kl}a_l)^{\cdot}\, f(0) + \frac{4\pi}{3}\dot{Q}^{kl}(\partial_l f)(0)$,

$$\mathcal{J}^\mu_{(2)} = \mathcal{I}^\mu_{(2)}(x,\mathbf{Q}) + \mathcal{I}^\mu_{(1)}(x,\mathbf{P}), \tag{108}$$

where the dipole charge $P^k = -\frac{2}{3}Q^{kl}a_l$. Therefore the Fermi frame quadrupole has a dipole component in the modified Fermi frame.

D Integrals, distributions and the proof of Theorem 2.

First we compute the following integral over two–sphere:

$$S_{klm} = \int_{S^2(r)} \left(\frac{x}{r}\right)^{2k}\left(\frac{y}{r}\right)^{2l}\left(\frac{z}{r}\right)^{2m} d\sigma, \quad k,l,m = 0,1,2,\ldots, \tag{109}$$

where $r = (x^2+y^2+z^2)^{1/2}$ is the radius of the two–sphere, $d\sigma = \sin\theta d\theta d\phi$, r,θ,ϕ are the spherical coordinates.

Proposition 1. One has

$$S_{klm} = \frac{(2k-1)!!(2l-1)!!(2m-1)!!}{(2k+2l+2m+1)!!}4\pi.$$

Proof. Let $A>0$. We use

$$\int_{\mathbf{R}^3} x^{2k}e^{-Ax^2}y^{2l}e^{-Ay^2}z^{2m}e^{-Az^2}dxdydz = \int_0^\infty r^{2(k+l+m)}e^{-Ar^2}r^2S_{klm}dr,$$

$$\int_{\mathbf{R}} x^{2k}e^{-Ax^2}dx = (-d/dA)^k\int_{\mathbf{R}} e^{-Ax^2}dx = \frac{(2k-1)!!}{2^k}\pi^{1/2}A^{-k-1/2}$$

and the analogous formulae for y,z and r. Q.E.D.

Corollary.

$$\int_{S^2(r)} x_{i_1} x_{i_2} \dots x_{i_{2n}} d\sigma = \frac{4\pi r^{2n}}{(2n+1)!} \sum_{\lambda \in \Pi_{2n}} \delta_{i_{\lambda(1)} i_{\lambda(2)}} \dots \delta_{i_{\lambda(2n-1)} i_{\lambda(2n)}}. \tag{110}$$

Proof. We get nonzero results only if $i_s = 1$ in $2k$ cases, $i_s = 2$ in $2l$ cases, $i_s = 3$ in $2m$ cases, $k + l + m = n$. Then the left hand side gives $S_{klm} r^{2n}$ and the sum gives the factor $(2k-1)!!(2l-1)!!(2m-1)!!2^n n!$. Next we use (109). Q.E.D.

Proposition 2. Let $Q^{i_1 \dots i_n}$ be a symmetric traceless tensor, $S^{j_1 \dots j_m}$ be any tensor. We set $Qx \cdots x = Q^{i_1 \dots i_n} x_{i_1} \cdots x_{i_n}$ and similarly for S. Then for $m = n$

$$\int_{S^2(r)} (Qx \cdots x)(Sx \cdots x)\, d\sigma = Q^{i_1 \dots i_n} S_{i_1 \dots i_n} \frac{4\pi (n!)^2 2^n}{(2n+1)!} r^{2n}, \tag{111}$$

while for $m < n$ the left hand side of (111) equals zero.

Proof. Let $m = n$. Then the left hand side of (111) is equal to $Q^{i_1 \dots i_n} S^{i_{n+1} \dots i_{2n}}$ multiplied by (110) and summed over all i_k. We may first sum over all i_k and then over $\lambda \in \Pi_{2n}$. In such a case, due to the traceless condition for $\mathbf{Q}$, nonzero terms in the sum over λ are obtained if for each $k = 1, \dots, n$ one of elements $\lambda(2k-1)$, $\lambda(2k)$ belongs to $\{1, \dots, n\}$ and the other one belongs to $\{n+1, \dots, 2n\}$. We get

$$\frac{4\pi r^{2n}}{(2n+1)!} 2^n n! \sum_{\rho \in \Pi_n} \sum_{i,j} Q^{i_1 \dots i_n} S^{j_1 \dots j_n} \delta_{i_{\rho(1)} j_1} \dots \delta_{i_{\rho(n)} j_n},$$

which due to the symmetry of Q gives the right hand side of (111). For $m < n$ a similar arguments show that all terms vanish (we use the traceless condition for Q if $m+n$ is even and the antisymmetry of the expression under the integral if $m+n$ is odd). Q.E.D.

Let $n = 0, 1, 2, \dots$, $\mathbf{Q}$ be a tracelless symmetric tensor of rank n, f be an even analytic function of r near 0 (f is a constant for $n = 0$). We set

$$D_{n\mathbf{Q}f} = f \frac{Qx \cdots x}{r^{2n+1}}$$

and define $\mathcal{F}$ as the linear span of all $D_{n\mathbf{Q}f}$. The elements $F \in \mathcal{F}$ become distributions if for any test function ρ we set

$$< F, \rho >= \int_0^\infty dr \int_{S^2(r)} r^2 F \rho d\sigma = \lim_{R \to 0^+} \int_{\mathbf{R}^3 \setminus K(0,R)} F \rho d^3 x. \tag{112}$$

That is well defined because setting $F = D_{n\mathbf{Q}f}$ and using

$$\rho = \sum_{k=0}^{n-1} \frac{1}{k!} \rho_{i_1 \dots i_k} x^{i_1} \dots x^{i_k} + O(r^n), \tag{113}$$

$$\rho_{i_1 \dots i_k} = (\partial_{i_1} \dots \partial_{i_k} \rho)(0), \tag{114}$$

we obtain

$$\int_{S^2(r)} r^2 F \rho d\sigma = \left[\sum_{k=0}^{n-1} \frac{1}{k!} \frac{f}{r^{2n-1}} \int_{S^2(r)} (Qx \cdots x)(\rho x \cdots x) d\sigma \right] + O(r) = O(r),$$

since all the integrals over $S^2(r)$ vanish due to $k < n$ and Proposition 2.

For $F \in \mathcal{F}$ let ∂_i denote the partial derivative in the sense of distributions, ∂_i^C - the partial derivative as function,

$$\partial_i^R F = \partial_i F - \partial_i^C F. \tag{115}$$

We define $\mathbf{l}_i$ as tensor of rank 1 such that $(l_i)_j = \delta_{ij}$.

Proposition 3.

$$x_i D_{n\mathbf{Q}f} = D_{n+1, \mathbf{l}_i \vee \mathbf{Q}, r^2 f} + \frac{n}{2n+1} D_{n-1, \mathbf{l}_i \rfloor \mathbf{Q}, f},$$

$$\partial_i^C D_{n\mathbf{Q}f} = D_{n+1, \mathbf{l}_i \vee \mathbf{Q}, -(2n+1)f + rf_{,r}} + \frac{n}{2n+1} D_{n-1, \mathbf{l}_i \rfloor \mathbf{Q}, f_{,r}/r}.$$

Proof. We use $x_i(Qx \cdots x) = (\mathbf{l}_i \vee \mathbf{Q})(x \cdots x) + \frac{n}{2n+1} r^2 Q_i$, where $Q_i = Q_{ii_2 \dots i_n} x^{i_2} \dots x^{i_n} = (\mathbf{l}_i \rfloor \mathbf{Q})(x \dots x)$. Q.E.D.

Thus for $F \in \mathcal{F}$, $\partial_i^C F$ and $\partial_i^R F$ are distributions.

Proposition 4. For $F \in \mathcal{F}$ and a test function ρ

$$< \partial_i^R F, \rho > = \lim_{R \to 0^+} \int_{S^2(R)} F \rho x_i R \, d\sigma.$$

Proof. Due to (112)

$$< \partial_i F, \rho > = - < F, \partial_i \rho > = \lim_{R \to 0^+} \int_{\mathbf{R}^3 \setminus K(0,R)} (\partial_i^C F) \rho d^3x - \lim_{R \to 0^+} \int_{\mathbf{R}^3 \setminus K(0,R)} \partial_i^C (F\rho) d^3x,$$

where the first term equals $< \partial_i^C F, \rho >$. Setting $(G_i)_j = F \rho \delta_{ij}$ and using (115), one gets

$$< \partial_i^R F, \rho > = - \lim_{R \to 0^+} \int_{\mathbf{R}^3 \setminus K(0,R)} \mathrm{div} G_i \, d^3x$$

$$= \lim_{R\to 0^+} \int_{S^2(R)} (G_i)_j \frac{x^j}{R} R^2 d\sigma = \lim_{R\to 0^+} \int_{S^2(R)} F\rho x_i R\, d\sigma. \quad \text{Q.E.D.}$$

Proposition 5.

$$\partial_i^R D_{n\mathbf{Q}f} = [\lim_{r\to 0} f(r)](-1)^{n-1}\frac{4\pi n}{(2n+1)!!} Q^{iI}\partial_I \delta^{(3)},$$

where $I = (i_2, \dots i_n)$, $\partial_I = \partial_{i_2} \dots \partial_{i_n}$.

Proof. Using Proposition 4 for $F = D_{n\mathbf{Q}f}$, (113), setting $c = \lim_{r\to 0} f(r)$, $(\rho_i)^{i_1\dots i_k i_{k+1}} = \rho^{i_1\dots i_k}\delta_i^{i_{k+1}}$ and using Proposition 2, we get

$$\rho_i(x\cdots x) = \rho^{i_1\dots i_k} x_{i_1}\cdots x_{i_k} x_i,$$

$$< \partial_i^R D_{n\mathbf{Q}f}, \rho >= \sum_{k=0}^{n-1} \frac{1}{k!} \lim_{R\to 0^+} \int_{S^2(R)} f(R) \frac{\rho_i(x\cdots x) Q(x\cdots x)}{R^{2n}} d\sigma =$$

$$\frac{c}{(n-1)!} Q^{i i_2\dots i_n}\rho_{i_2\dots i_n} \frac{4\pi (n!)^2 2^n}{(2n+1)!}.$$

Moreover, we use $< \partial_I \delta^{(3)}, \rho >= (-1)^{n-1}\rho_{i_2\dots i_n}$ (cf. (114)) and $\frac{(2n+1)!}{n! 2^n} = (2n+1)!!$ Q.E.D.

Proposition 6. Assuming $f \in L_{n0}$ (cf. Section 4) and setting $g = fr^n$ (which is regular at 0), we obtain

$$X_{k(n)}(f, \mathbf{Q}) = D_{n+1, \mathbf{l}_k \vee \mathbf{Q}, (2n+1)g - rg_{,r}} + \frac{n+1}{2n+1} D_{n-1, \mathbf{l}_k \rfloor \mathbf{Q}, g_{,r}/r},$$

$$Y_{k(n)}(f, \mathbf{Q}) = -D_{n, \mathbf{l}_k \times \mathbf{Q}, g/\varphi}.$$

Proof. By a direct computation. Q.E.D.

Proposition 7.

$$\partial_i^R X_{k(n)}(r^{-n}, \mathbf{Q}) = (-1)^n \frac{4\pi(2n+1)}{(2n+3)!!} \times$$

$$\times [\delta_{ik} Q^J \partial_J \delta^{(3)} + n Q_i{}^I \partial_{kI}\delta^{(3)} - \frac{n(n-1)}{2n+1} Q_{ik}{}^L \partial^m{}_{mL}\delta^{(3)} - \frac{2n}{2n+1} Q_k{}^I \partial_{iI}\delta^{(3)}],$$

$$\partial_i^R X_{k(n)}(r^{-(n-2)}, \mathbf{Q}) = (-1)^n \frac{4\pi(n-1)}{(2n-1)!!} Q_{ik}{}^L \partial_L \delta^{(3)},$$

$$\partial_i^R X_{k(n)}(f, \mathbf{Q}) = 0 \text{ if } \lim_{r\to 0}(fr^{n-2}) = 0,$$

$$\partial_i^R Y_{k(n)}(r^{-n}, \mathbf{Q}) = (-1)^n \frac{4\pi}{(2n+1)!!}[(n-1)\epsilon_k{}^{ma} Q_{mi}{}^L \partial_{aL}\delta^{(3)} + \epsilon_{ik}{}^m Q_m{}^I \partial_I \delta^{(3)}],$$

$$\partial_i^R Y_{k(n)}(f, \mathbf{Q}) = 0 \text{ if } \lim_{r\to 0}(fr^n) = 0.$$

Proof. We use Proposition 6 and Proposition 5. Q.E.D.

Remark. In the case of $n = 0, 1$ some equations related to the first equation of Proposition 7 were presented in (6)–(7) of [4].

Next, we obtain that in addition to (25), (26)–(27), (34)–(35), the operations div, curl_φ and # acting on the fields $\mathbf{X}$ and $\mathbf{Y}$ yield the following distribution parts:

Proposition 8.

$$\mathrm{div}^R \mathbf{X}_{(n)}(r^{-n}, \mathbf{Q}) = (-1)^n \frac{4\pi(n+1)}{(2n+1)!!} Q^J \partial_J \delta^{(3)},$$

$$[\mathrm{curl}_\varphi^R \mathbf{X}_{(n)}(r^{-n}, \mathbf{Q})]_k = (-1)^n \frac{4\pi n}{(2n+1)!!} \epsilon_{ki}{}^j Q^{iI} \partial_{jI} \delta^{(3)},$$

$$[\mathbf{X}_{(n)}(r^{-n}, \mathbf{Q})^{\#R}]_k = (-1)^n \frac{4\pi n(n^2-1)}{(2n+1)!!} Q_{ki}{}^L b^i \partial_L \delta^{(3)},$$

$$\left[\mathrm{curl}_\varphi^R \mathbf{Y}_{(n)}(r^{-n}, \mathbf{Q})\right]_k = (-1)^n \frac{4\pi(n+1)}{(2n+1)!!} Q_k{}^I \partial_I \delta^{(3)},$$

$$\mathrm{div}^R \mathbf{Y}_{(n)}(r^{-n}, \mathbf{Q}) = \mathbf{Y}_{(n)}(r^{-n}, \mathbf{Q})^{\#R} = 0,$$

while for f such that $\lim_{r\to 0}(fr^n) = 0$ the distribution parts vanish.

Proof. We use Proposition 7 and notice that the operator $d/d\tau$ appearing in # (cf. (15)) gives no additional terms. Q.E.D.

Let us notice that $\psi_0 = D_{011} = \frac{1}{r}$ is the classical monopole potential, $-\Delta\psi_0 = 4\pi\delta^{(3)}$. By induction one has $Q^I \partial_I \psi_0 = (-1)^n (2n-1)!! D_{n\mathbf{Q}1}$ (no distribution parts! - cf. Proposition 5) and therefore $\psi_n = D_{n\mathbf{Q}1}$ satisfies

$$-\Delta\psi_n = (-1)^n \frac{4\pi}{(2n-1)!!} Q^I \partial_I \delta^{(3)}$$

(cf. (41)) and ψ_n is the classical 2^n–pole potential. Using (115) and Propositions 3,5 and 6, we obtain that the corresponding electric field is given by

$$\mathcal{D}_n^{cl} = -\mathrm{grad}\psi_n, \quad (\mathcal{D}_n^{cl})_k = X_{k(n)}(r^{-n}, \mathbf{Q}) + (\mathcal{D}_d)_k, \tag{116}$$

where

$$(\mathcal{D}_d)_k = (-1)^n \frac{4\pi n}{(2n+1)!!} Q_k{}^I \partial_I \delta^{(3)}$$

(cf. (45) and Remark 2 in Section 5). After some computations one obtains

$$\mathrm{div}\mathcal{D}_d = (-1)^n \frac{4\pi n}{(2n+1)!!} Q^J \partial_J \delta^{(3)}, \tag{117}$$

$$(\mathrm{curl}_\varphi \mathcal{D}_d)_k = (-1)^{n+1} \frac{4\pi n}{(2n+1)!!} \epsilon_{ki}{}^j Q^{iI} \partial_{jI} \delta^{(3)}, \tag{118}$$

$$(\mathcal{D}_d^{\#})_k = (-1)^n \frac{4\pi n}{(2n+1)!!} \dot{Q}_k{}^I \partial_I \delta^{(3)}. \tag{119}$$

Proof of Theorem 2. Let us first omit the terms containing $\delta^{(3)}$ and its derivatives. Then $\mathcal{D}_{(l)}$ and $\mathcal{B}_{(l)}$ consist of $\mathbf{X}$ and $\mathbf{Y}$ fields and therefore (cf. (25)) $\mathrm{div}\mathcal{D}_{(l)} = \mathrm{div}\mathcal{B}_{(l)} = 0$, which proves (17)–(18). Moreover, applying curl_φ to (47)–(48), we obtain

$$\mathrm{curl}_\varphi \mathcal{B}_{(k+1)} = \mathcal{D}_{(k)}{}^{\#}, \quad k = -n-2, -n, -n+2, \ldots, \tag{120}$$

$$-\,\mathrm{curl}_\varphi \mathcal{D}_{(k+1)} = \mathcal{B}_{(k)}{}^{\#}, \quad k = -n-1, -n+1, -n+3, \ldots. \tag{121}$$

But one has also

$$\mathrm{curl}_\varphi \mathcal{D}_{(-n-2)} = 0 \tag{122}$$

(we use (27) and $h_{(n)}[P_n(z)] = 0$). Inserting (43)–(44) into the both sides of (19)–(20) and using (120)–(122), one proves that (19)–(20) are fulfilled.

Let us notice that $\delta^{(3)}$ and its derivatives appear in the sources (41)–(42), in $\mathcal{D}_d$ and also when acting by div, curl_φ or # on $\mathbf{X}_{(k)}(r^{-k}, \mathbf{P})$, $\mathbf{Y}_{(k)}(r^{-k}, \mathbf{P})$. Using (32)–(35), one shows that $\mathbf{X}_{(k)}(r^{-k}, \mathbf{P})$, $\mathbf{Y}_{(k)}(r^{-k}, \mathbf{P})$ appear only in $\mathcal{D}_{(-n-2)}$ which contains $\mathbf{X}_{(n)}(r^{-n}, \mathbf{Q})$ and in $\mathcal{B}_{(-n-1)}$ which contains

$$-\frac{n^2-1}{2n+1} \mathbf{Y}_{(n-1)}(r^{-(n-1)}, \mathbf{b}\rfloor\mathbf{Q}) - \mathbf{Y}_{(n)}(r^{-n}, \dot{\mathbf{Q}}).$$

Therefore $\mathrm{div}\mathcal{D}$ contains additionally

$$\mathrm{div}\mathcal{D}_d + \mathrm{div}^R \mathbf{X}_n(r^{-n}, \mathbf{Q}) = (-1)^n \frac{4\pi}{(2n-1)!!} Q^J \partial_J \delta^{(3)} = \mathcal{J}^0,$$

div$\mathcal{B}$ has no additional terms, $\mathcal{D}^{\#} - \mathrm{curl}_\varphi \mathcal{B}$ contains additionally (cf. Proposition 8, (119) and (42))

$$\mathcal{D}_d{}^{\#} + \mathbf{X}_{(n)}(r^{-n}, \mathbf{Q})^{\# R} + \frac{n^2-1}{2n+1} \mathrm{curl}_\varphi^R \mathbf{Y}_{(n-1)}(r^{-(n-1)}, \mathbf{b} \rfloor \mathbf{Q})$$

$$+\mathrm{curl}_\varphi^R \mathbf{Y}_{(n)}(r^{-n}, \dot{\mathbf{Q}}) = -\mathcal{J},$$

$\mathcal{B}^{\#} + \mathrm{curl}_\varphi \mathcal{D}$ contains additionally $\mathrm{curl}_\varphi \mathcal{D}_d + \mathrm{curl}_\varphi^R \mathbf{X}_{(n)}(r^{-n}, \mathbf{Q}) = 0$, hence the Maxwell equations are satisfied also in the distributional sense. The last statement of Theorem 2 follows from the properties of #, $\mathrm{curl}_\varphi^{-1}$ and $\mathbf{X}$ and $\mathbf{Y}$ fields. Q.E.D.

Acknowledgements

We thank Dr Marcin Kościelecki for fruitful discussions.

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